On the origin of the cosmological constant

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Abstract

Under the assumption that the cosmological constant vanishes in the true ground state with lowest possible energy density, we argue that the observed small but finite vacuum-like energy density can be explained if we consider a theory with two or more degenerate perturbative vacua, which are unstable due to quantum tunneling, and if we still live in one of such states. An example is given making use of the topological vacua in non-Abelian gauge theories.

There are *two* cosmological constant (Λ) problems today. One is the older problem why the vacuum energy density is vanishingly small, or why the intrinsic cosmological term cancels with accumulation of zero-point energy in quantum field theory almost exactly. Observationally, the vacuum energy density $\rho_{\rm v} = 3M_G^2\Lambda$ is no larger than the critical density $\rho_{cr0} = 4 \times 10^{-47}{\rm GeV}^4 = (3{\rm meV})^4$ today, where $M_G = M_{Pl}/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV is the reduced Planck scale. On the other hand, since a natural cutoff scale of zero-point fluctuations of each quantum field is the Planck scale, we would expect $\langle\rho_{\rm v}\rangle \simeq M_G^4$ from them, which is larger than the observational constraint by a factor of 10^{120} [1].

The second, newer problem is that the above miraculous cancellation does not seem to work in a perfect manner, that is, there is increasing evidence that a finite positive component still remains in the vacuum-like energy density and that our Universe is in a stage of accelerated expansion now. For example, the analysis of SNIa data shows that the probability that we live in a universe with $\Lambda = 0$ is less than one percent [2].

Historically, a number of solutions have been proposed about the first problem [1, 3]. Among them, the quantum cosmological approach is based on the Euclidean path integral of the wave function of the universe [4]. It has been claimed that such a path integral is dominated by the de Sitter instanton solution proportional to $\exp(\frac{3\pi}{G\Lambda})$ and hence it is likely that the cosmological constant vanishes [5]. Coleman further incorporated fluctuations of the spacetime topology in terms of the "wormhole" configurations and found a double exponential dependence [6]. One should note, however, that the expectation values obtained in these approaches should be regarded as giving an average over the time in the history of the universe [1]. So they may not necessarily be related with the values we observe today. We may rather interpret it as predicting a vanishing cosmological constant in some ground state where the universe spends most of the time in history. [1] More recently, a number of higher dimensional models have been proposed in which the maximally symmetric solution of the three-brane must have a vanishing four-dimensional cosmological constant [9]. Since our Universe has not settled in a maximally symmetric state, it is difficult to understand implications of these results on current values of cosmological parameters in our Universe, but we are tempted to interpret them as predicting a vanishing cosmological constant again in some ultimate ground state.

Here we argue the possible origin of the small but finite cosmological constant without introducing any small numbers. We solve the second problem under the assumption that the first one is solved in the true ground state by arguing that we have not fallen into that state. Other proposed solutions to the second problem, such as quintessence [10] (see also [11] for earlier work) or meV-scale false vacuum energy [12], are also based on such a hypothesis.

Our starting point is that the energy eigenvalue of the true ground state of a theory with two or more degenerate perturbative vacua, which cannot be transformed from one another without costing energy, is smaller than that of a quasi-ground state localized around one of these states in field space by an

[1] We must also point out problems with Euclidean formulation of quantum gravity, namely, positive-nondefiniteness of the Euclidean action, the ambiguity of the signature of rotation [7], and the negativity of the phase of the saddle point solution [8].

exponentially small amount. Here, by perturbative vacua we mean a state with the lowest energy density without taking possible tunneling effect to another perturbative vacuum into account. Our hypothesis is that the cosmological constant vanishes *not* in these degenerate perturbative vacua but in the absolute ground state with quantum tunneling effects taken into account, because there are both classical and quantum contributions to the cosmological constant and what we observe is their sum.

For illustration let us first consider an abstract field theory model whose perturbative vacuum states are classified into two distinct categories labeled by $|+\rangle$ and $|-\rangle$ with $\langle +|-\rangle = 0$ at the lowest order. We also assume that, although the transition from $|+\rangle$ to $|-\rangle$ is classically forbidden, there is an instanton solution which describes quantum tunneling from $|+\rangle$ to $|-\rangle$ and vice versa. By nature the instanton is localized in space and (Euclidean) time with a finite Euclidean action S_0. Then the true ground state, $|S\rangle$, with this tunneling effect taken into account, is given by the symmetric superposition of $|+\rangle$ and $|-\rangle$, namely,

$$|S\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \tag{1}$$

where we have assumed that $|+\rangle$ and $|-\rangle$ are normalized.

Now we evolve $|S\rangle$ with Euclidean time T to calculate $\langle S|e^{-\mathcal{H}T}|S\rangle$ by summing up contributions of instantons and anti-instantons as

$$\begin{aligned}\langle S|e^{-\mathcal{H}T}|S\rangle &= \frac{1}{2}\left(\langle +|e^{-\mathcal{H}T}|+\rangle + \langle -|e^{-\mathcal{H}T}|-\rangle + \langle +|e^{-\mathcal{H}T}|-\rangle + \langle -|e^{-\mathcal{H}T}|+\rangle\right) \\ &= e^{-\rho_0 VT}\sum_{j=0}^{\infty}\frac{1}{(2j)!}\left(kVTe^{-S_0}\right)^{2j} + e^{-\rho_0 VT}\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}\left(kVTe^{-S_0}\right)^{2j+1} \\ &= \exp\left(-\rho_0 VT + kVTe^{-S_0}\right), \end{aligned}\tag{2}$$

where $\mathcal{H}$ is the Hamiltonian, $k \equiv m^4$ is a positive constant and V represents spatial volume [13]. Here ρ_0 is the energy density of the perturbative vacua, $|+\rangle$ and $|-\rangle$, which are presumably translationally invariant. Thus the energy density of the true ground state $|S\rangle$ is given by

$$\rho_S = \rho_0 - m^4 e^{-S_0}. \tag{3}$$

It is this energy density that vanishes under our hypothesis. This in turn implies that we find a nonvanishing vacuum energy density

$$\rho_0 = m^4 e^{-S_0} \tag{4}$$

in either of the perturbative vacuum states, $|+\rangle$ or $|-\rangle$.

Thus if our Universe is in one of the perturbative vacuum states because it is too young to be relaxed into the true ground state $|S\rangle$, we observe a nonvanishing vacuum energy density (4) today. Since the tunneling rate per unit volume per unit time is given by $\Gamma \simeq m^4 e^{-2S_0}$ apart from a prefactor of order unity [14], the requirement that there should be no transition in the horizon volume in the cosmic age reads

$$\Gamma H_0^{-4} \simeq 9M_G^4/m^4 \lesssim 1, \tag{5}$$

where $H_0^2 \cong \rho_0/(3M_G^2)$ is the current Hubble parameter squared. We therefore find that, if the parameters satisfy $m \gtrsim M_G$ and $S_0 = 120\ln 10 + 4\ln(m/M_G)$, we can account for the observed small value of the cosmological constant without introducing any small numbers.

So far is a generic study to generate an exponentially small difference in energy density using a theory with degenerate perturbative vacua whose real ground state is given by their superposition. Next in order to see how this mechanism may be implemented in a more specific theory with this property, let us consider a famous example of an SU(N) ($N \geq 2$) gauge theory whose perturbative vacuum states are classified in terms of the winding number n and denoted by $|n\rangle$ [15, 16]. States with different winding numbers cannot be transformed from each other by a continuous gauge transformation [16] and there is an energy barrier between them. Let us concentrate on the simplest case with $N = 2$ hereafter. An instanton solution [17], which describes quantum tunneling from one perturbative vacuum to another with the change of the winding number $\Delta n = 1$, can be expressed as

$$A_\mu(x) = \frac{2R^2\eta_{a\mu\nu}(x_\nu - y_\nu)\sigma^a}{(x-y)^2\left[(x-y)^2 + R^2\right]}, \tag{6}$$

where $\eta_{a\mu\nu}$ is the 't Hooft symbol [18], σ^a is the Pauli matrix and R is the size of the instanton. Here y_ν represents spacetime coordinates at its center. Thanks to the translational and the scale invariance, the Euclidean action does not depend on these quantities, namely,

$$S_0 = \frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} = \frac{8\pi^2}{g^2}, \tag{7}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu$ is the field strength and g is the gauge coupling constant.

The true ground state in the presence of quantum tunneling is given by an infinite sum of these $|n\rangle$ states as

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta}|n\rangle, \tag{8}$$

where θ is a real parameter [15, 16]. One can easily find that this state is a real eigenstate of the Hamiltonian $\mathcal{H}$ in terms of the following calculation based on the dilute instanton approximation [15].

$$\langle\theta'|\, e^{-\mathcal{H}T}\,|\theta\rangle = \exp\left(2KVTe^{-S_0}\cos\theta\right)\delta\left(\theta-\theta'\right), \tag{9}$$

where $KVTe^{-S_0}$ represents contribution of a single instanton or anti-instanton. Here K is a positive constant and VT again represents spacetime volume.

Then the above equality (9) clearly shows that each θ-vacuum $|\theta\rangle$ has a different energy density than the perturbative vacuum $|n\rangle$ by

$$\Delta\rho = -2Ke^{-S_0}\cos\theta. \tag{10}$$

Apparently the $\theta = 0$ vacuum has the lowest energy (and no CP violation), but one cannot conclude that this is the only vacuum state, because other θ-vacua are also stable against gauge-invariant perturbations [15, 16] and transition between different θ-vacua is impossible even when instantons are incorporated into calculation as is seen in (9). Since different θ-vacua have different energy density they behave differently in the presence of gravity, and we must choose some particular value of θ where we assume the cosmological vanishes. This choice should ultimately be made together with the solution of the older cosmological constant problem. At present, however, since there is no completely satisfactory solution to this problem, we simply normalize the vacuum energy density to vanish in the most natural CP-symmetric $\theta = 0$ vacuum state here. We point out that the wormhole mechanism [6] is an example of proposed solutions to the older cosmological constant problem which selects $\theta = 0$ vacuum as the "ground" state with $\Lambda = 0$ where the universe presumably spends most of the time [19]. With this normalization of Λ, the vacuum energy density in each perturbative vacua is found to be

$$\langle n|\rho_{\rm v}|n\rangle = 2Ke^{-S_0}. \tag{11}$$

In fact, the factor K, which is expressed as

$$K = \frac{\pi^2}{4}\left(\frac{8\pi}{g^2(\mu)}\right)^4\int\frac{dR}{R^5}\exp\left(-\frac{8\pi^2}{g^2(\mu)} + \frac{22}{3}\ln(\mu R) + 6.998435\right), \tag{12}$$

in the case of the pure SU(2) gauge theory [20], is divergent due to the contribution of arbitrary large instantons. Here μ is a renormalization scale. In order to obtain a physical cutoff scale let us introduce an SU(2) doublet scalar field Φ with a potential $V[\Phi] = \lambda(|\Phi|^2 - M^2/2)^2/2$, following 't Hooft [20]. For $M \neq 0$, the solution (6) is no longer an exact one, but one could find an approximate solution, a constrained instanton [21], with the following properties.
(i) For $RM \lesssim 1$, the solution is given by (6) and

$$|\Phi(x)| = \left(\frac{(x-y)^2}{(x-y)^2+R^2}\right)^{1/2}\frac{M}{\sqrt{2}}, \tag{13}$$

for $(x-y)^2 \lesssim M^{-2}$.
(ii) For larger $(x-y)^2$, the solution rapidly approaches to the vacuum values with

$$|\Phi(x)| - \frac{M}{\sqrt{2}} \sim e^{-\sqrt{\lambda}M|x-y|}, \qquad A_\mu(x) \sim e^{-gM|x-y|}. \tag{14}$$

(iii) The Euclidean action is finite and approximately given by

$$S = \frac{8\pi^2}{g^2} + \pi^2 R^2 M^2. \tag{15}$$

Using (15) in (12), the integral is now given by

$$K = \frac{\pi^2}{4}\left(\frac{8\pi}{g^2}\right)^4 \int \frac{dR}{R^5} \exp\left[-\frac{8\pi^2}{g^2} - \pi^2 R^2 M^2 + \frac{43}{6}\ln\left(\frac{RM}{\sqrt{2}}\right) + 6.759189\right], \tag{16}$$

and we find,

$$\rho_{\rm v} \cong M^4 \left(\frac{8\pi}{g^2}\right)^4 e^{-\frac{8\pi^2}{g^2}}, \tag{17}$$

$$\Gamma \simeq M^4 \left(\frac{8\pi}{g^2}\right)^4 e^{-\frac{16\pi^2}{g^2}}. \tag{18}$$

Demanding that $\rho_{\rm v} = 10^{-120} M_G^4$ and that the tunneling rate in the current horizon should be smaller than unity in the cosmic age, $\Gamma H_0^{-4} \lesssim 1$, we find

$$\frac{\pi}{\alpha} + 2\ln\alpha = 60\ln 10 + 2\ln\left(\frac{M}{M_G}\right), \tag{19}$$

$$M \gtrsim \alpha M_G, \tag{20}$$

where $\alpha \equiv g^2/(4\pi)$ is the coupling strength at the energy scale $M/\sqrt{2}$. If the inequality (20) is marginally satisfied, we find $\alpha = 1/44.4$ and $M = 5 \times 10^{16}$GeV. If, on the other hand, we take $M = M_G$ so that the cutoff scale of instanton is identical to the presumed field-theory cutoff, we find $\alpha = 1/47$.

Thus if our Universe happened to be created in a state with some specific winding number and remains there up until now, we would observe a nonvanishing vacuum energy density (17). Although θ-vacuum is the real ground state of the theory, there is no *a priori* reason that the Universe is created in this state. In fact, in the possibly chaotic initial state of the early universe [22], there may well be a small domain where the scalar field has a nonvanishing expectation value with a constant SU(2) phase and A^a_μ vanishes. If such a region is exponentially stretched by cosmological inflation [23] and our Universe is contained in it, the state of our Universe would be more like the perturbative vacuum $|n=0\rangle$ than some θ-vacuum. Then it is not surprising that we observe a nonvanishing vacuum energy density (17) today. Thus our mechanism can be naturally implemented in the chaotic inflation scenario [22].

Note that the Hubble parameter in the observable regime of inflation, H, is constrained as $H/(2\pi) \lesssim 4\times 10^{13}$GeV so that the tensor-induced anisotropy of cosmic microwave background radiation [24] satisfies $\delta T/T \lesssim 10^{-5}$ [25]. Hence the amplitude of quantum fluctuations generated along the phase direction of the fields during inflation is much smaller than M, and it does not affect the realization of the state $|n=0\rangle$. Furthermore gravitational effects are negligibly small for the instanton configuration even during inflation because $H \ll M$. We also note that thermal transition to a state with a different winding number is suppressed since the reheat temperature after inflation, T_R, is typically much smaller than M [23]. In fact, to avoid overproduction of gravitinos, it should satisfy $T_R < 10^{12}$GeV [26]. Hence we have only to worry about quantum transition (18) as we have already done.

In summary, we have pointed out that in a field theory with two (or more) degenerate perturbative vacua, the vacuum energy density of the true ground state is smaller than that in a perturbative vacua by an exponentially small amount if quantum tunneling between degenerate vacua is possible, and that this may be utilized to explain the observed small value of the cosmological constant without introducing any small quantities.

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